The part of "A.14: Optimization" that is about a function with one variable

maximum & minimum

Saying that a function $y = f(x)$ achieves a **maximum** at a point x^* in its domain means $f(x^*) \geq f(x)$ for all x in its domain.

Saying that a function $y = f(x)$ achieves a **minimum** at a point x^* in its domain means $f(x^*) \leq f(x)$ for all x in its domain.

Consider a smooth function $y = f(x)$ whose domain is a closed interval, [a, b].

When f achieves a maximum or a minimum at a point x^* in the interior of [a, b], the corresponding tangent line is horizontal [in other words, in the graph for f, the function's curve is flat at $(x^*, f(x^*))$]. So the slope of the function is zero.

As a consequence:

Let x^* be a point in the interior of [a, b]. If f achieves a maximum at x^*, then $f'(x^*) = 0$.

Hence, if we want to find a point in the interior of [a, b] where f achieves a maximum or a minimum, it is helpful to start by finding where the slope of the function is zero (that is, where the first derivative is zero).

(Note: $f'(x^*) = 0$ is commonly called the "first-order condition" for a maximum or a minimum.)

Looking at this observation in a different way: The points in the interior of [a, b] where the first derivative does not equal zero can be eliminated from consideration (since they are points where maxima or minima couldn't be achieved).

The second derivative can also be helpful, since:

1) When f achieves a maximum at a point x^* in the interior of [a, b], the second derivative at x^* has to satisfy $f''(x^*) \leq 0$;

2) When f achieves a maximum at a point x^* in the interior of [a, b], the second derivative at x^* has to satisfy $f''(x^*) \geq 0$.

(Note: The first inequality is called the "second-order condition" for a maximum; the second inequality is commonly called the "second-order condition" for a minimum)

Looking at this observation in a different way:

1) If we want to find a point where f achieves a maximum, any point x^* in the interior of [a, b] where $f'(x^*) = 0$ and $f''(x^*) > 0$ can be eliminated from consideration (since it is a point where a maximum couldn't be achieved);

2) If we want to find a point where f achieves a minimum, any point in the interior of [a, b] where $f'(x^*) = 0$ and $f''(x^*) < 0$ can be eliminated from consideration (since it is a point where a minimum couldn't be achieved).

The material that follows states some additional results about optimization that involve second derivatives.

local maximum & local minimum

Saying that a function $y = f(x)$ achieves a **local maximum** at a point x^* in its domain means there exists an open interval (c, d) such that $f(x^*) \geq f(x)$ for all x contained in the intersection of (c, d) and the domain of f.

Saying that a function $y = f(x)$ achieves a **local minimum** at a point x^* in its domain means there exists an open interval (c, d) such that $f(x^*) \leq f(x)$ for all x contained in the intersection of (c, d) and the domain of f.

Once again, consider a smooth function $y = f(x)$ whose domain is a closed interval, [a, b].

When the "first-order condition" is satisfied at a point in the interior of [a, b], the second derivative can sometimes tell us that we have found a local maximum or a local minimum:

1) if $f'(x^*) = 0$ <u>and</u> $f''(x^*) < 0$, then x^* is a local maximum;

2) if $f'(x^*) = 0$ <u>and</u> $f''(x^*) > 0$, then x^* is a local minimum.

Suppose $f''(x) \leq 0$ for all x in its domain. Let x^* be a point in the interior of [a, b]. f achieves a maximum at x^* if and only if $f'(x^*) = 0$.

Suppose $f''(x) \geq 0$ for all x in its domain. Let x^* be a point in the interior of [a, b]. f achieves a minimum at x^* if and only if $f'(x^*) = 0$.

If $f''(x) < 0$ for all x in its domain, then there is a unique point in the domain where f achieves a maximum.

If $f''(x) > 0$ for all x in its domain, then there is a unique point in the domain where f achieves a minimum.